Phase transitions in Abelian lattice gauge theories

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Abstract. We study the phase transition in the $U(1)$ lattice gauge theory using the Wilson–
Polyakov line as the order parameter. The Wilson–Polyakov line remains very small at strong
coupling and becomes non-zero at weak coupling, signalling a confinement-to-deconfinement phase
transition. The decondensation of monopole loops is responsible for this phase transition. A finite
size scaling analysis of the susceptibility of the Wilson line gives a ratio for γ/ν which is quite
close to the corresponding value in the three-dimensional planar model. A scaling behaviour of
the monopole loop distribution function is also established at the point of the second-order phase
transition. A measurement of the plaquette susceptibility at the transition point shows that it does
not scale with the four-dimensional volume as is expected of a first-order bulk transition.

1. Introduction

Lattice gauge theories (LGTs) at non-zero temperatures have been the focus of many
investigations in recent years. Their study enables us to make non-perturbative predictions for
the high-temperature properties of gauge theories. LGTs are expected to clarify many issues
about finite-temperature gauge theories: the nature of the high-temperature phase, the order
of the phase transition, and the relevant elementary excitations at high temperatures. The
pioneering work in [1] was the first non-perturbative demonstration that quarks are deconfined
at high temperatures. This calculation is done in the strong coupling limit of the $SU(2)$
LGT. Early Monte Carlo simulations [2] provided further support for the existence of this
phase transition. Since then, there have been many studies of the thermodynamic properties
of the $SU(2)$ and the $SU(3)$ LGTs [3] which have provided valuable insights into the high-
temperature behaviour of gauge theories.

For a lattice model to throw light on a field theory, we have to take the zero lattice spacing
limit, $a \to 0$, in a way such that observed quantities on the lattice go over to finite values in
the continuum. A necessary prerequisite for a continuum theory to exist is the presence of a
second-order phase transition in the coupling space of the theory. To have a finite-temperature
system, periodic boundary conditions are imposed in the Euclidean time direction with period
$\beta(\frac{1}{T})$; the passage to the continuum limit also requires that the temperature $T = \frac{1}{N_\tau a}$ (N_τ is
the extent of the lattice in the temporal direction) remains finite. This further entails taking
the limit $N_\tau \to \infty$, along with the limit $a \to 0$. Again, the second-order phase transition
must persist in the limit $N_\tau \to \infty$ for us to define a finite-temperature field theory. In order to
achieve conditions allowing the study of thermodynamic properties, lattice studies are made
with a fixed N_τ for large spatial lattices ($N_\sigma \gg N_\tau$), and then one tries to explore the behaviour
on lattices having larger N_τ. In this paper, we take a step towards addressing these issues for

the $U(1)$ LGT. More precisely, we study the $U(1)$ LGT on asymmetric lattices for which $N_\sigma \gg N_\tau$. N_σ is the lattice size in the spatial direction and N_τ is the lattice size in the temporal direction. Fixing N_τ allows a richer phase diagram for the $U(1)$ LGT, and this phase structure is relevant to define a finite-temperature field theory. That the $U(1)$ LGT has a first-order bulk transition might make it appear that there is no way to define a continuum field theory in its phase diagram. However, there have been some recent claims for a second-order phase transition in an extended version of the $U(1)$ LGT with a monopole chemical potential term [15]. The study of compact $U(1)$ LGT using different lattice topologies [16] has also shown that second-order transitions are possible in this model. Also, new phase transitions can appear as a function of N_τ and, if these transitions are of second order, we are still left with the possibility of defining non-trivial field theories, even at finite temperature, for the $U(1)$ LGT. There are several other reasons why we have embarked on a study of this simple model. Firstly, the bulk properties of this model have been inferred from simulations on large symmetric lattices ($N_\sigma = N_\tau$) in [4]. In these simulations, a phase transition is observed from a confining phase to a deconfining phase. Although there was some controversy about the order of the phase transition, later simulations on large lattices strongly suggested that the bulk transition is of first order [5]. The mechanism of this bulk transition is quite well understood in terms of monopole loop excitations [11]. These monopole loops are topological objects which arise because of the periodicity properties of the action. The bulk system exists in two phases, a confining phase in which the monopole currents condense causing complete Meissner effect, and a deconfining phase in which the monopoles are too heavy to have any physical effect. Since the monopoles can wind more effectively around a finite lattice, it is interesting to see how they can alter the properties of the system. Recently [6], there have been some studies which showed that these considerations could affect the order of the phase transition. Secondly, the critical behaviour of lattice gauge theories at high temperatures can be understood from analogous behaviour of three-dimensional spin models. The strong coupling analysis in [1] shows that the partition function of the $SU(2)$ LGT can be rewritten as a three-dimensional spin model with a global $Z(2)$ symmetry. This symmetry arises because finite-temperature gauge systems have an additional global symmetry arising from the periodic boundary conditions in the temporal direction. For systems with a gauge symmetry group G, this global symmetry group consists of elements belonging to the centre of the gauge group. The deconfining phase corresponds to the symmetry-broken phase and the confining phase corresponds to the symmetric phase. Furthermore, as emphasized in [7], the order of the phase transition in four dimensions is expected to be dictated by the universality classes present in three-dimensional spin models having this global symmetry. These expectations have been borne out in studies of the $SU(2)$ [8] and the $SU(3)$ LGTs [9] in which one observes a second-order Ising-like and a first-order $Z(3)$-like phase transition, respectively. A similar question can be raised in the much simpler $U(1)$ LGT. Unlike the non-Abelian $SU(N)$ LGTs, which have a discrete centre subgroup, $(Z(N))$, the Abelian $U(1)$ LGT has a continuous centre subgroup which is identical to the group itself. The role of the $U(1)$ group on the confinement to deconfinement transition is also of some interest from the point of view of the Abelian dominance hypothesis for confinement which holds that a $U(1)$ subgroup controls the non-perturbative dynamics of non-Abelian gauge theories [17]. Finally, as the $U(1)$ LGT already has a bulk phase transition unlike the $SU(2)$ and $SU(3)$ LGTs, there is the question of the interplay between this transition with the expected finite-temperature transition. The study of the $U(1)$ LGT on asymmetric lattices allows a richer phase structure than the one seen on symmetric lattices, and it is an interesting issue to decide whether the new phase transitions are similar or any different from the bulk phase transitions. This matter has been recently examined in the context of $SU(2)$ LGTs using a mixed action; it was found in [10] that the bulk transition and

the finite-temperature transition may coincide, making it difficult to distinguish one from the other. A major difference between a bulk transition and a finite-temperature transition, which can in principle distinguish the two transitions, is the movement of the transition point with the temporal lattice size. For non-Abelian gauge theories, a physically relevant finite-temperature transition should move towards the weak coupling region in a manner which is specified by the beta function of an asymptotically free gauge theory. With these notions in mind, we would like to embark on our present study of the $U(1)$ LGT.

We will mainly consider the Wilson action [18] for the $U(1)$ LGT, which is given by

$$S = \beta \sum_{n\mu > \nu} \cos(\theta(n\mu\nu)). \tag{1}$$

The $\theta(n\mu\nu)$ are the usual oriented plaquette variables:

$$\theta(n\mu\nu) = \theta(n\mu) + \theta(n+\mu\nu) - \theta(n+\nu\mu) - \theta(n\nu). \tag{2}$$

The link variables $\theta(n\mu)$ can take values from $-\pi$ to π. As mentioned before, the properties of this model at zero temperature are well known. There is a transition at $\beta \approx 1.0$ that is caused by a decondensation of monopole currents (the bulk transition on a 16^4 lattices has been located precisely in [21] and occurs at $\beta = 1.016$). These monopole currents are defined on the dual lattice by counting the number of Dirac strings entering or leaving a three-dimensional cube on the original lattice [19]. The monopole density on a link $(\star l)$ of the dual lattice is defined as

$$\rho(\star l) = \frac{-1}{2\pi} \sum_{p \in c} \bar{\theta}(p). \tag{3}$$

$\bar{\theta}(p)$ is extracted from $\theta(p)$ by expressing it as

$$\theta(p) = \bar{\theta}(p) + 2\pi n(p) \tag{4}$$

so that it takes values from $-\pi$ to π. In the above expression, the monopole current is defined on the links $\star l$ of the dual lattice which are dual to the cubes c in the original lattice. As the monopoles form closed loops on the dual lattice, it is more convenient to measure the total perimeter density of the monopole current loops. The perimeter density is given by

$$\rho = \frac{1}{N_l} \sum_l \rho(l). \tag{5}$$

N_l is the total number of links in the lattice. One observable which is of special relevance to our analysis is the Wilson–Polyakov line (henceforth called the Wilson line), that is defined as

$$L(\vec{n}) = \prod_{n_0=0}^{N_\tau} \exp(i\theta(\vec{n} + n_0\hat{4}\hat{4})). \tag{6}$$

This observable is gauge invariant and has played a crucial role in studies of the deconfinement transition in $SU(N)$ LGTs. The centre symmetry that emerges at finite temperature is the transformation which multiplies by a constant phase all the time-like links emanating from some fixed time slice, namely

$$\exp(i\theta(\vec{n}\hat{4})) \to \exp(i\alpha)\exp(i\theta(\vec{n}\hat{4})). \tag{7}$$

Although the action is invariant under this transformation, the Wilson line transforms as

$$L(\vec{n}) \to \exp(i\alpha)L(\vec{n}). \tag{8}$$

Hence, a non-zero expectation value of the Wilson line signals a spontaneous breakdown of the centre symmetry. The Wilson line is given the usual physical interpretation by writing it in the form

$$\langle L(\vec{n}) \rangle = \exp(-\beta F_q(\vec{n})). \tag{9}$$

It measures the free energy $(F_q(\vec{n}))$ of a static charge in a heat bath at a temperature β^{-1}. Hence, a non-zero value of $\langle L(\vec{n}) \rangle$ indicates deconfinement of static charges whereas a zero value indicates confinement. We point out a subtlety in this representation, that has also been noticed before in the context of the $SU(2)$ LGT [13]. The left-hand side is in general a complex quantity and hence cannot be given a free-energy interpretation. One way of avoiding this problem is to work with the correlation function of two Wilson lines which is always a non-negative quantity. Apart from the Wilson line, we have also studied the susceptibility of the Wilson line which is defined as

$$\chi_l = N_\sigma^3 (\langle \vec{s}^2 \rangle - \langle |\vec{s}| \rangle^2). \tag{10}$$

The 'spin' variable $\vec{s}$ is constructed in equation (14). The peak in the Wilson line susceptibility can be used to locate the phase transition.

A related observable that is also relevant is the plaquette susceptibility, defined as

$$\chi_p = 6N_\sigma^3 N_\tau \sum_p (\langle P^2 \rangle - \langle P \rangle^2). \tag{11}$$

In the above equation, P is the average plaquette density in the system. The peak in the plaquette susceptibility can also be used to locate the phase transition. According to the finite size scaling theory, the susceptibility of an observable, which blows up at a phase transition point in the infinite volume limit, is expected to scale with the system size as

$$\chi_p^{cr} = N^\alpha. \tag{12}$$

Here N is the size of the system. For a first-order transition, α equals d, the dimensionality of the system; for a second-order transition, α equals γ/ν, where γ and ν are the exponents for the susceptibility and the correlation length near the transition, which are given by

$$\chi = (T - T_c)^{-\gamma} \qquad \xi = (T - T_c)^{-\nu}. \tag{13}$$

We will use the above observables to study the phase transition in the Abelian LGT.

This paper is organized as follows. Section 2 contains the results of our numerical investigations of this model. In section 3 we make some comments on the mixed action $U(1)$ LGT and compare and contrast it with that of the of mixed action $SU(2)$ LGT. In section 4 we summarize our conclusions.

2. Numerical results

In this section we present our numerical analysis of the model. The system is mimicked at finite temperature by working on an asymmetric lattice ($N_\sigma \gg N_\tau$) with periodic boundary conditions in the Euclidean time direction. We first briefly describe the numerical procedure that we adopted to obtain our results. The Metropolis algorithm was used to generate successive Monte Carlo configurations. A new link variable θ' was generated from the old one θ by adding a number randomly chosen in the range $(-\alpha, \alpha)$ with uniform probability. The value of α was tuned to get an acceptance of 50%. Care was taken so that the link variables remained in the range $(-\pi, \pi)$. Simulations were performed on temporal lattices with $N_\tau = 2, 3, 4$. The observables that were measured are: the monopole density, the Wilson line, the plaquette density, the Wilson line susceptibility, and the plaquette susceptibility. As the Wilson line is complex (it is a phase with modulus equal to one), we measure its real and imaginary parts separately. If we simply measure the average value of the real or the imaginary parts, the result will always be equal to zero because phase transitions are impossible on finite system as the tunnelling between the degenerate states always restores the symmetry. A rigorous way of studying symmetry breaking is to study the average value of the Wilson line in the presence

of a small symmetry breaking external field, and then take the limit of zero field after taking the large volume limit. A simpler prescription, that is often employed in studying continuous spin models, is to study the root mean square value of the order parameter, and this is how we will proceed. The observable that we have measured as an indicator of spontaneous symmetry breaking is $\sqrt{\langle s \rangle}$; s is defined as

$$s = \operatorname{Re} L^2 + \operatorname{Im} L^2. \tag{14}$$

Here $\operatorname{Re} L$ and $\operatorname{Im} L$ refer to the average of the real and imaginary parts of the Wilson line respectively. A simple strong coupling analysis (valid for β small) yields the following effective action for the Wilson lines:

$$S_{\text{eff}} = 2\left(\frac{\beta}{2}\right)^{N_\tau} \sum_{\vec{n}\vec{n}'} \cos(\theta(\vec{n}) - \theta(\vec{n}')). \tag{15}$$

N_τ is the temporal extent of the lattice and the $\theta(\vec{n})$ variables are the sums of the phases of all the time-like links at the spatial point $\vec{n}$. This is the action for the three-dimensional planar model which is known to have an order–disorder transition at $\beta_{cr} = 0.454$. For an $N_\tau = 2$ lattice, this gives the critical coupling to be approximately 0.95. Thus we expect our lattice model to have a phase transition at $\beta \approx 0.9$. This strong coupling argument is valid only if the phase transition takes place in the strong coupling regime. Nevertheless, the strong coupling approximation provides a simple way of seeing how a three-dimensional spin model emerges from the four-dimensional gauge theory.

A more direct way of seeing the appearance of an effective three-dimensional spin model, without using a strong coupling approximation, is to use the dual representation of the four-dimensional $U(1)$ LGT. The dual representation is given by

$$Z = \int D\phi \sum_{m_\mu(r)} \exp -\frac{1}{2\beta} \sum_{n\mu\nu} (\partial_\mu \phi_\nu - \partial_\nu \phi_\mu)^2 + 2\pi i \sum_n m_\mu(n)\phi_\mu(n). \tag{16}$$

This describes a gas of closed monopole loops ($m_\mu(r)$) which interact by a four-dimensional Coulomb potential. This system has a phase transition which takes place as a result of the competition between the energy and the entropy of the monopole loops. The fields $\phi_\mu(r)$ can be integrated out to give

$$Z = \sum_{m_\mu(r)} \exp\left(-2\pi^2\beta \sum_{nn'} m_\mu(n)G_{\mu\nu}(n-n')m_\nu(n')\right). \tag{17}$$

At non-zero temperatures, the time direction is finite and the Green function satisfies periodic boundary conditions. The four-dimensional Green function can be rewritten as

$$G_{\mu\nu}(r-r') \approx T\tilde{G}_{\mu\nu}(\vec{r}-\vec{r}'). \tag{18}$$

Here $\tilde{G}_{\mu\nu}(\vec{r}-\vec{r}')$ is the three-dimensional Green function. As the gas is finite in one direction, one again has a gas of monopole loops which now effectively interact with a three-dimensional Coulomb interaction. From the point of view of the effective three-dimensional planar model, the monopole loops behave like vortex lines. The entropy of large loops in three dimensions is smaller than in the four-dimensional case and this shifts the transition. The order of the transition cannot be deduced from these energy arguments and has to be determined by doing a finite size scaling analysis. We show in figure 1 the variation of $\sqrt{\langle s \rangle}$ with β on a $6^3\,2$ lattice. The observable $\sqrt{\langle s(\vec{n}) \rangle}$ is close to zero at small β and rises smoothly across the critical value. The $U(1)$ monopole density variation is shown in figure 2. There is a fall in the monopole density across the transition which coincides with the rise in the order parameter. In both cases, the variation is quite gradual and we would suspect that we are in the vicinity
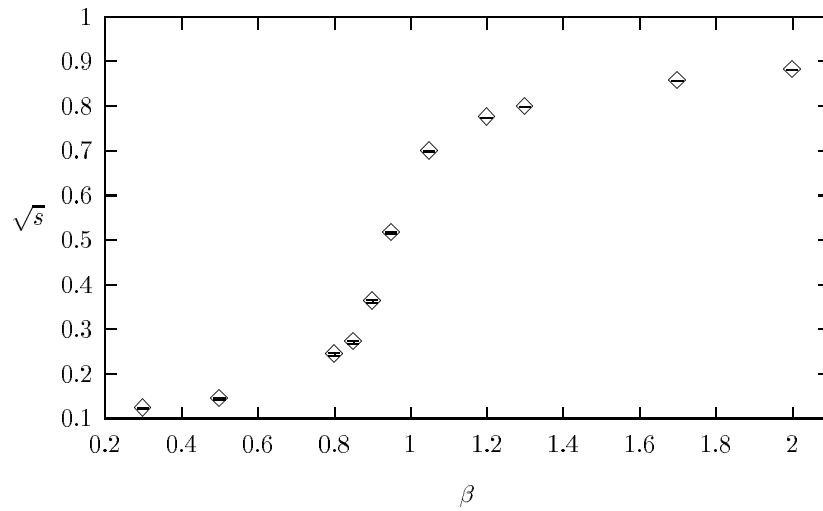
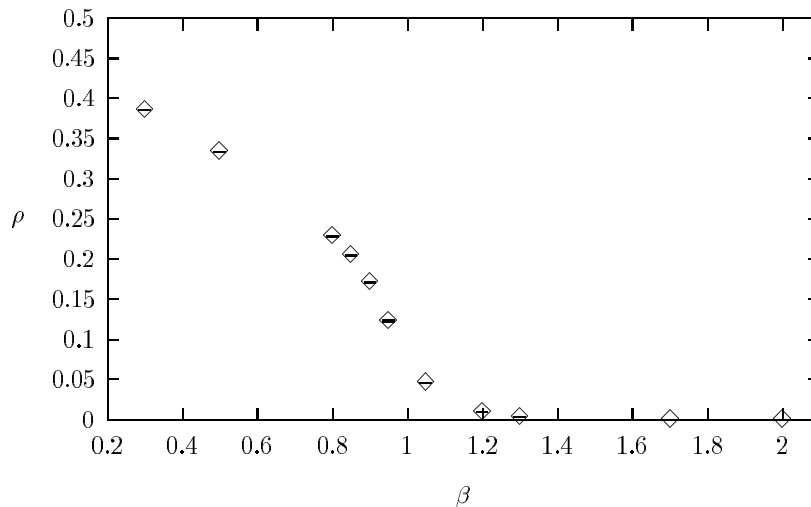**Figure 1.** Variation of the order parameter on a $6^3 2$ lattice.

Figure 2. Monopole density on a $6^3 2$ lattice.

of a second-order transition. This is further suggested by the gradual rise in the plaquette expectation value (see figure 3). To determine the order of the transition, we perform a finite size scaling analysis of the susceptibility of the order parameter (χ_l). We have done the finite size scaling study on lattices of temporal extent $N_\tau = 3$ and 4. For this method to work, it is crucial that we are very close to the pseudo-critical point corresponding to the lattice that we are working on. The histogram method is used to extrapolate observables from one value to a nearby neighbouring value. The pseudo-critical point is located in this way by looking at the peak   in the susceptibility of the order parameter. The behaviour of the susceptibility near the transition on 6, 9, 12 and 16 sized spatial lattices (keeping the temporal extent fixed at $N_\tau = 3$) is shown in figure 4. These results were obtained after 200 000 measurements on $N_\sigma = 6, 9$ lattices while 150 000 measurements were made on the $N_\sigma = 12$ lattice and
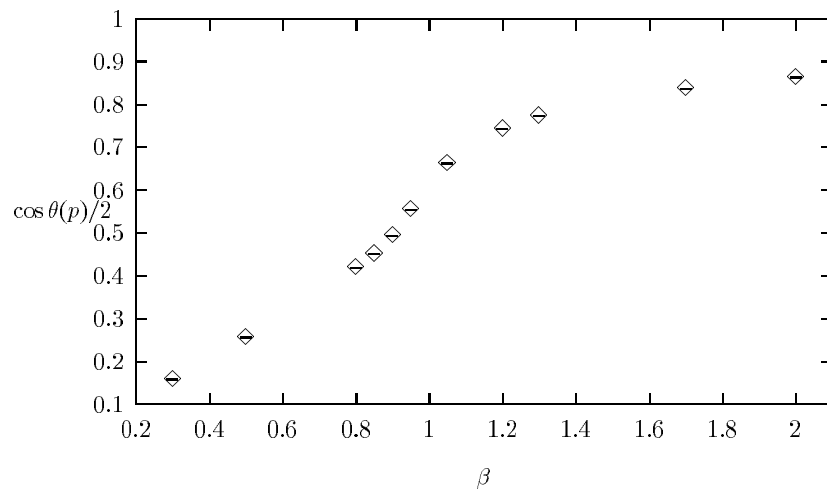
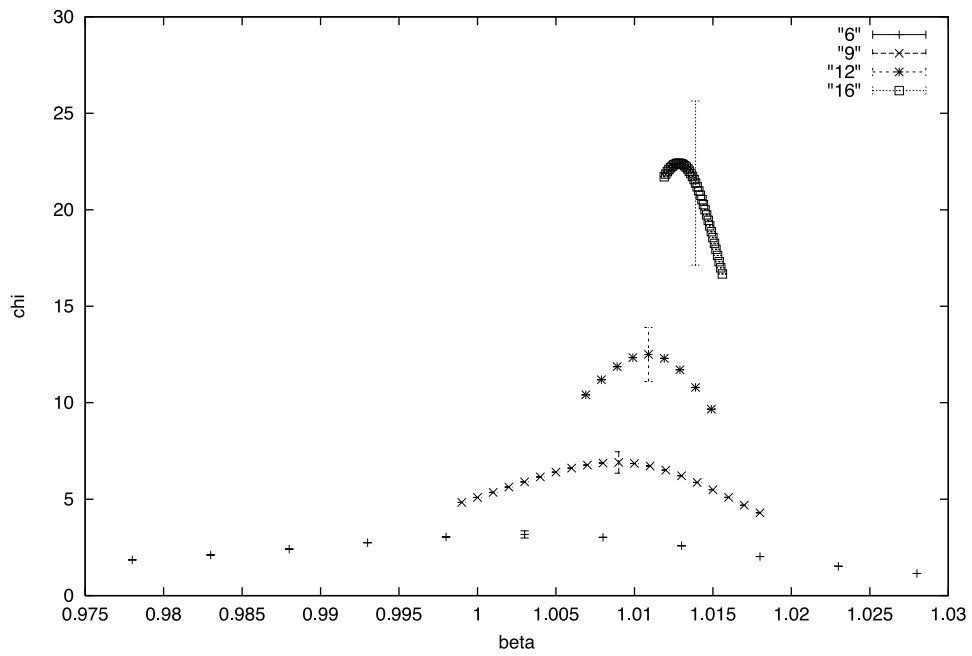**Figure 3.** Plaquette expectation value on a $6^3\,2$ lattice.

Figure 4. Susceptibility of the order parameter near the transition on 6, 9, 12 and 16 size spatial lattices.

$100\,000$ measurements were made on the $N_\sigma = 16$ lattice. All the measurements were made after ignoring the first $50\,000$ Monte Carlo iterations. The errors were estimated by binning the data. The finite size scaling theory predicts
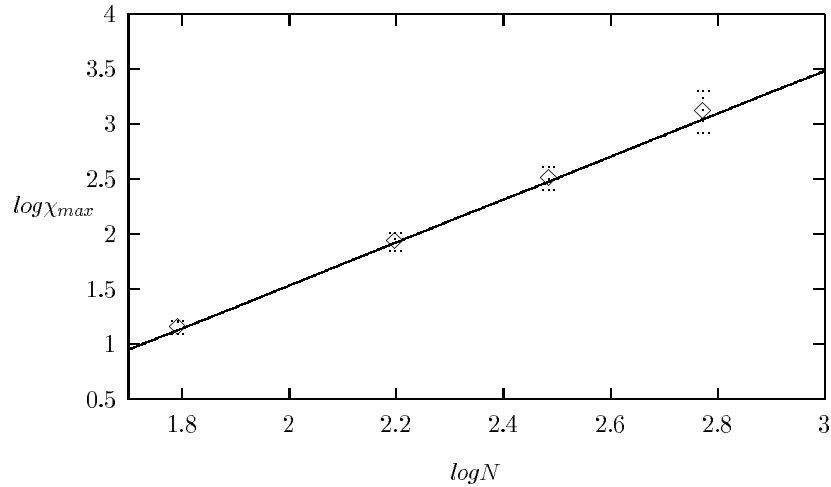
$$\chi \approx N_\sigma^d \tag{19}$$

Figure 5. A fit of the maximum value of the logarithm of the susceptibility to ln N.

for a first-order phase transition; d being the effective dimension of the system (in this case $d = 3$); for a second-order transition, the prediction is

$$\chi \approx N_\sigma^{\frac{\gamma}{\nu}}. \tag{20}$$

γ and ν are the exponents in equation (13). The peaks in the susceptiblity were fit to an N_σ^α dependence and a good fit is obtained (figure 5). From our fit we determine γ/ν to be 1.95 with an error of 0.13. For the three-dimensional planar model, which is the effective spin model with which we would like to compare our results, the ratio $\gamma/\nu = 1.97$.

We have performed another test for the order of the phase transition by studying the distribution of the monopole loops. Since the monopole loops, which now behave as vortex lines, are responsible for driving the phase transition, they should also exhibit a scaling behaviour at the point where the transition becomes second order. At the point of the second-order phase transition, there are fluctuations on all length scales and the monopole loops come in all sizes and shapes. At the transition point, the monopole loop distribution function, $p(l)$, should scale as

$$p(l) = \frac{1}{l^\tau} \tag{21}$$

with an exponent τ which is independent of the volume. The function $p(l)$ is defined as the probability of finding a loop of length l at a site. In order to define $p(l)$ we have to decide what we are to do when loops intersect, which is the generic case. Whenever a loop intersects, we trace the smallest possible path of the loop that returns to its starting point. In this way, the loops that we measure cannot be broken down to any smaller loops and they are essentially non-intersecting. We have calculated $p(l)$ for various lattices at their pseudo-critical points and we find that $p(l)$ can be nicely fitted to the above form. The value of τ that we get (using data on a $N_\sigma = 16$ lattice) is 2.27 ± 0.003. Unfortunately, we are not able to compare it with a theoretical calculation. Nevertheless, it is interesting that the topological excitations, in this case the monopoles, exhibit a scaling dependence at the transition point. Figure 6 shows $p(l)$ for loops of length l up to 14 on different lattices. The plot shows that they fall on a straight line independent of the volume. The slight distortions for the $N_\sigma = 6$ lattice are due to finite size corrections when the loop length is larger than the lattice size. A study of larger loop
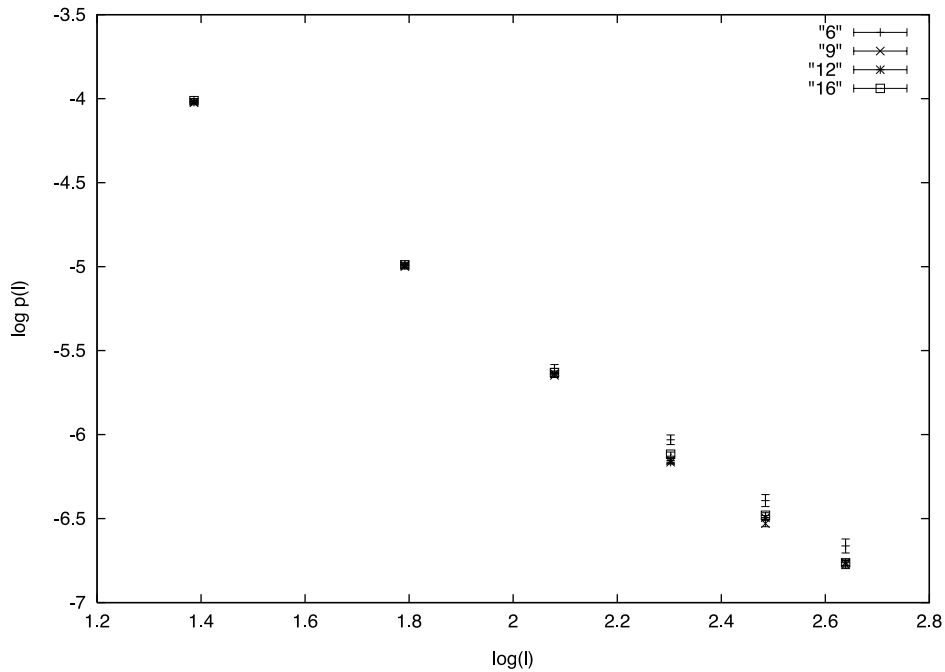
Figure 6. Scaling of the monopole loop distribution function $p(l)$ with loop length (l). The graph shows the logarithm of $p(l)$ as a function of the logarithm of l.

lengths can also be made but the algorithm that we have used to count the loops slows down when the loop lengths become very large, and so we have not studied larger loops. Away from the critical point, $p(l)$ either falls rapidly to zero (in the deconfining phase) or remains constant (in the confining phase). A similar study of the monopole loop distribution function was made in [20] for the Abelian projected monopoles in the $SU(2)$ theory near the continuum limit.

Now we say a few words on the location of this transition as compared with the location of the bulk transition. The transition point (which is located by looking at the peak in the susceptibility of the order parameter) shifts as a function of N_τ as

N_τ	2	3	4
β_{cr}	0.9297(3)	1.012(2)	1.032(2).

The above critical values are those obtained on N_σ of 16, 16 and 12 respectively. The bulk transition on a 16^4 lattice was located at $\beta = 1.016$ in [5]. Since the bulk transition is known to be of first order [5], and the transition that we have observed is of second order, we are observing a change in the order of the phase transition as a function of N_τ. The nature of the transition that we have observed can be further studied by monitoring the behaviour of the plaquette susceptibility which was defined in equation (11). The behaviour of the plaquette susceptibility for a $6^3\,3$ and a $12^3\,3$ lattice is shown in figure 7. This graph clearly shows that the plaquette susceptibility does not scale with the four-dimensional volume as is expected of a first-order bulk transition.

The above observations show that the transition that we have observed (on an $N_\tau = 3$ lattice) is a deconfinement transition whose scaling behaviour is quite distinct from that of the bulk transition. The order of the transition as seen by the Wilson line is a second-order transition with the ratio γ/ν which has approximately the same value as that in the three-
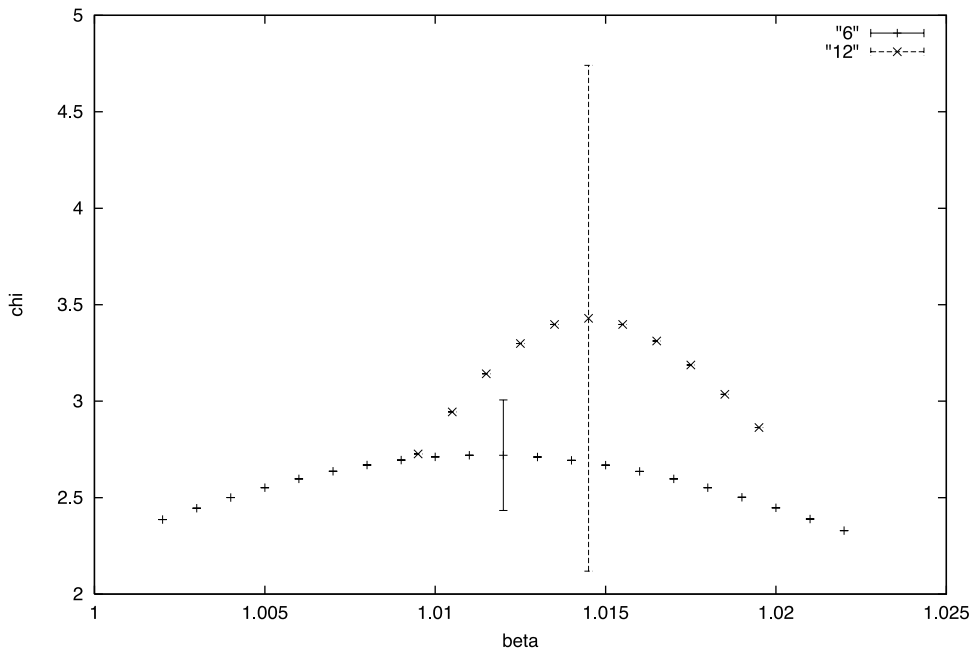
Figure 7. The scaling of the plaquette susceptibility near the phase transition. This is shown on $6^3\,3$ and $12^3\,3$ lattices.

dimensional planar model. Since the plaquette susceptibility does not scale as a first-order bulk transition, we are not observing the bulk transition at the point where we are observing the finite-temperature transition. The finite size scaling analysis can be repeated on an $N_\tau = 4$ lattice and the peak in the susceptibility of the order parameter as a function of volume is shown in figure 8. This scaling again suggests a second-order transition with a value for γ/ν which is again in good agreement with that of the three-dimensional planar model.

We now place our results in the context of some recent developments. The studies in [10] considered the finite-temperature properties of the mixed action $SU(2)$ LGT which is defined by

$$S = \left(\frac{\beta_f}{2}\right)\sum_p \mathrm{tr}_f\ U(p) + \left(\frac{\beta_a}{3}\right)\sum_p \mathrm{tr}_a U(p). \tag{22}$$

This model is known to have lines of first-order bulk transitions in the β_f, β_a plane [12]. It was found in [10] that the deconfinement transition of the pure $SU(2)$ LGT, which is of second order, continues into the β_f, β_a plane and joins the line of first-order bulk transitions. Based on this observation in [10], a possibility was considered where there is either only a bulk or a finite-temperature transition in the $SU(2)$ LGT. On the other hand, there may also be a very small but finite separation between the two transitions which cannot be resolved on the lattice sizes used in the simulations. In our case, the relevant coupling space is the β, N_τ^{-1} plane. Our simulations of the $U(1)$ LGT show that there is a deconfinement transition which is of second order and that the plaquette susceptibility does not scale as a first-order bulk transition at the transition point. The transition is shown to shift from its bulk value as a function of N_τ but this shift is very small, though still discernible. We also place our results in the light of the expectations of Svetitsky and Yaffe [7]. According to their general arguments, the critical behaviour of the $U(1)$ LGT at finite temperature is expected to fall in the same universality
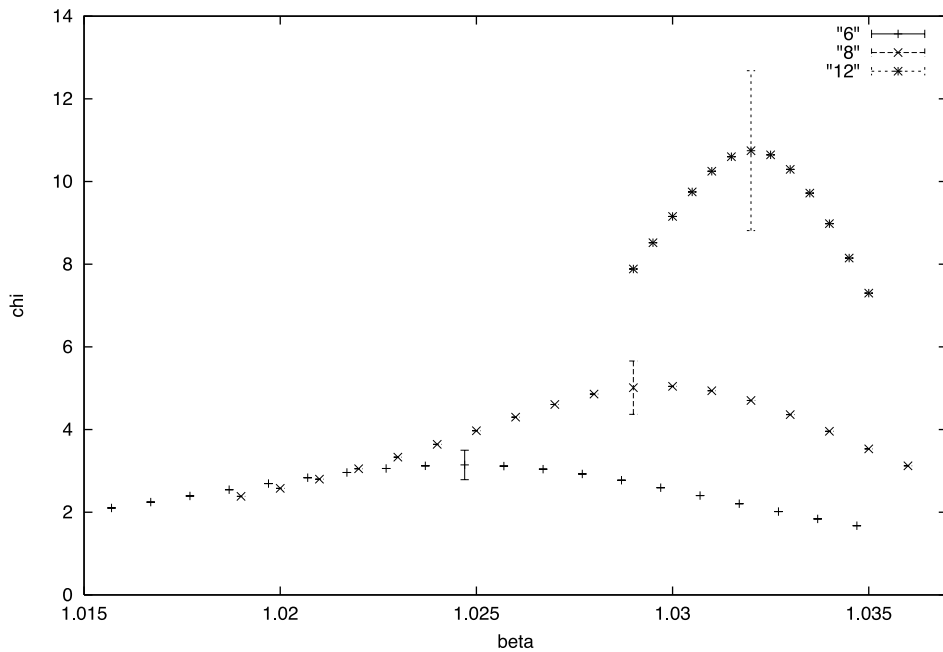
Figure 8. A finite size study of the susceptibility of the order parameter on an $N_\tau = 4$ lattice.

class as that of the three-dimensional planar model, which is known to have a second-order phase transition. Our finite size scaling analysis on the $N_\tau = 3$ lattice definitely rules out a first-order phase transition and indicates that the ratio γ/ν has approximately the same value as in the three-dimensional planar model.

We conclude this section with a proposal for the phase diagram of the $U(1)$ LGT as a function of N_τ. In the bulk system (which corresponds to taking the limit $N_\sigma = N_\tau$ and $N_\sigma \to \infty$), there is the monopole-driven phase transition. On asymmetric lattices of very small temporal extent, we might expect the bulk system to look like a three-dimensional system. On asymmetric lattices of very large temporal extent, we expect to see the four-dimensional bulk transition. Lattices of intermediate temporal extent will exhibit a complicated crossover from a four-dimensional system to a three-dimensional system. Our simulations on an $N_\tau = 3$ lattice show that there is a second-order phase transition which has the same value of γ/ν as in the three-dimensional planar model. This transition is a deconfinement transition as is indicated by the behaviour of the fundamental Wilson line. However, there is no bulk transition (as in the four-dimensional theory) on this lattice. Whether the transitions that have been observed in asymmetric lattices lead to a finite-temperature continuum limit will depend on their behaviour on lattices with larger N_τ.

3. Mixed action $U(1)$ LGT

Since lattice actions are, any way, not unique, we can always construct more complicated-looking lattice actions and examine their properties. A simple generalization of the action in section 1 is the mixed action which is defined by

$$S = \beta_1 \sum_p \cos(\theta(p)) + \beta_2 \sum_p \cos(2\theta(p)). \tag{23}$$

The two pieces of the above action are different from each other only in so far as their periodicity properties are concerned. In the naive continuum limit, $a \rightarrow 0$, the second term is like an irrelevant coupling and is not expected to change the long-distance properties of the theory. The zero temperature properties of this action have been studied in [14] and it has a rich phase structure of first- and second-order transitions. This system can also be studied at finite temperature just as we did for the $\beta_2 = 0$ theory. Again, from a simple strong coupling analysis of the mixed action $U(1)$ LGT, we get an effective theory of spins which is that of the mixed planar model:

$$S_{\text{eff}} = 2 \left(\frac{\beta_1}{2} \right)^{N_\tau} \sum_{\vec{n}\vec{n}'} \cos(\theta(\vec{n}) - \theta(\vec{n}')) + 2 \left(\frac{\beta_2}{2} \right)^{N_\tau} \sum_{\vec{n}\vec{n}'} \cos(2\theta(\vec{n}) - 2\theta(\vec{n}')). \tag{24}$$

Putting $\beta_1 = 0$ gives a three-dimensional planar model of spins, the only difference being in the periodicity properties of the action. Hence, the finite-temperature properties of the mixed model in the $\beta_1 = 0$ limit should be identical to those in the $\beta_2 = 0$ limit. In particular, the previous statements regarding the order of the transition will also be true in this limit. A surprising feature of the mixed planar model is that it posesses a region of first-order phase transitions for some values of β_2 [23]. This implies a similar region of first-order transitions in the mixed $U(1)$ LGT for a segment of β_2 values. The order of the finite-temperature transition changing in the direction of an irrelevant coupling has also been discussed in the context of the mixed action $SU(2)$ LGT [10].

From the above analysis, it is clear that there are many similarities between the $U(1)$ LGT and the mixed action $SU(2)$ LGT. We now show that the similarity also extends to the construction of the relevant order parameters in the two models. The mixed action $SU(2)$ LGT [12] is defined by

$$S = \frac{\beta_f}{2} \sum_p \text{tr}_f U(p) + \frac{\beta_a}{3} \sum_p \text{tr}_a U(p). \tag{25}$$

tr_f and tr_a denote the traces in the fundamental and the adjoint representations, respectively. The limit $\beta_a = 0$ describes an $SU(2)$ LGT and the limit $\beta_f = 0$ describes an $SO(3)$ LGT. The order parameter of the finite-temperature transition in the $SU(2)$ LGT is the Wilson line in the fundamental representation which is defined as

$$L_f(\vec{n}) = \text{Tr}_f \prod_{n_0=0}^{N_\tau} U(\vec{n} + n_0 \hat{4} \hat{4}). \tag{26}$$

In the $SO(3)$ LGT ($\beta_f = 0$), this observable is identically zero because of the following *local* $Z(2)$ symmetry:

$$U(\vec{n}\hat{4}) \rightarrow Z(\vec{n})U(\vec{n}\hat{4}). \tag{27}$$

$Z(n)$ can take the values $+1$ or -1 at any site. For the $SO(3)$ LGT, the appropriate order parameter is the Wilson line in the adjoint representation,

$$L_a(\vec{n}) = \text{Tr}_a \prod_{n_0=0}^{N_\tau} U(\vec{n} + n_0 \hat{4} \hat{4}) \tag{28}$$

which is invariant under the local $Z(2)$ transformation in equation (27). For the group $SU(2)$, L_f and L_a are related by

$$L_a(\vec{n}) = L_f(\vec{n})^2 - 1.0. \tag{29}$$

In the mixed action $U(1)$ LGT we are faced with a similar problem in defining the order parameter for the $\beta_1 = 0$ theory. In this limit, the Wilson line defined in section 1 is identically zero because of the following local symmetry:

$$\exp(i\theta(\vec{n}\hat{4})) \rightarrow Z(\vec{n})\exp(i\theta(\vec{n}\hat{4})). \tag{30}$$

The correct order parameter to use in this limit is

$$L_2(\vec{n}) = \prod_{n_0=0}^{N_\tau} \exp(i2\theta(\vec{n} + n_0\hat{4}\hat{4})) \tag{31}$$

which is analogous to the Wilson line in the adjoint representation of $SU(2)$. The relationship between $L_2(\vec{n})$ and $L(\vec{n})$ is $L_2(\vec{n}) = L(\vec{n})^2$.

Though there are similarities at the formal level, and in the phase structure of the mixed action Abelian and mixed action non-Abelian theories, there are, of course, many important differences between the two systems. An important difference is that, unlike in the Abelian LGT, the critical coupling in the $SU(2)$ LGT is expected to scale with N_τ according to the beta function of the Yang–Mills theory. There is strong evidence for this asymptotic scaling from simulations on very large temporal lattices [22].

The purpose of this section was only to indicate that many of the issues such as the mixing of the bulk and finite-temperature transitions which have been raised recently can all be explored in the much simpler mixed action Abelian LGT. Also, because the physical properties of this model are well understood in terms of the monopole excitations, this model may prove useful in investigating these issues.

4. Conclusions

In this paper we studied the $U(1)$ LGT using the Wilson line as the order parameter. We have found that there is a transition into a deconfining phase at large coupling which is driven by the decondensation of monopole loops. The monopole loops, however, effectively interact like the vortices in the three-dimensional planar model. The deconfining phase breaks the global $U(1)$ symmetry present in the theory. A finite size scaling analysis of the susceptibility of the Wilson line indicates that the transition is of second order (on lattices of temporal size $N_\tau = 3, 4$) with a ratio for γ/ν which has almost the same value as in the three-dimensional planar model. A scaling form of the distribution of the monopole loops was also established at the point of the second-order phase transition. This transition was also examined by studying the plaquette susceptibility at the transition point. The plaquette susceptibility (on a $N_\tau = 3$ lattice) does not scale as is expected of a first-order bulk transition. There is also a small shift in the transition point from the bulk value. We have also pointed out that many of the recent issues concerning the mixing of the bulk and finite-temperature transitions can also be raised in the Abelian LGT. Since the Abelian theories are well understood in terms of their monopole excitations, some of these issues can perhaps be clarified.

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